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Contact Stress from Asymptotic Reissner–Mindlin Plate Theory

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Introduction

PUBLISHED work that goes back decades shows conclusively that Euler–Bernoulli beam theory and Kirchhoff plate theory both fail, even qualitatively, to capture the behavior of transverse shear-stress resultants and contact stresses, such as when a beam or plate is being pressed against a flat surface. When the effects of transverse shear stresses are added to the theories, resulting in Timoshenko beam theory and Reissner–Mindlin plate theory, the accuracy of contact stresses and transverse shear stress resultants is improved.[1–3] In particular, the qualitatively incorrect result of discontinuous transverse shear-stress resultants is overcome, but the contact stresses remain discontinuous. However, in more recent work models that include degrees of freedom associated with transverse normal strain have been shown to yield continuous contact stresses.[4,5] Unfortunately, however, such models are considerably more complex than the classical theories just mentioned (for example, see Refs. 6 and 7).

Recently, beam[8] and plate[9] theories have been derived from three-dimensional elasticity using the variational-asymptotic method. In the plate theories the small parameter h/ℓ is used to reduce the dimensionality of the model, where h is the thickness of the plate and ℓ is the wavelength of deformation in the plane of the plate. A model that is derived in this manner and is valid to order $(h/\ell)^0$ has the form of a Kirchhoff or classical laminated plate theory, but it is not subject to the usual (and internally inconsistent) assumptions that the transverse normal stress is zero and that the normal line element remains straight, of constant length, and normal to the deformed plate. Instead, the normal line element deforms, even in isotropic plates, as a result of the Poisson effect. When all terms through order $(h/\ell)^2$ are kept, the resulting theory, whether for homogeneous, isotropic or laminated, composite plates, can be uniquely cast into the form of a Reissner–Mindlin plate theory. Additional warping of the normal line element occurs as a result of transverse shear effects. When all terms through order $(h/\ell)^2$ are kept in the asymptotic approximations for the three-dimensional field variables, accurate through-the-thickness distributions of all in-plane, transverse shear, and transverse normal strain and stress are obtained. These results are especially helpful in recovery of stress and strain components for laminated plates[10] or shells[11] and can be obtained without introducing any degrees of freedom beyond those of Reissner–Mindlin theory.

Thus, an equivalent single-layer theory for laminated composite plates, when consistently derived using the variational-asymptotic method, is capable of far more than just accurate prediction of global behavior. There is one additional feature of this type of theory that has been developed for plates, although not yet for beams. When the plate is loaded with tractions on its upper and lower surfaces and body forces, the potential of the applied loads will contain terms that arise because of the warping of the normal line element. Indeed, the purpose of this Note is to show that an asymptotically correct theory through order $(h/\ell)^2$, which includes such terms, is sufficient to obtain qualitatively correct transverse shear-stress resultants and transverse normal contact stresses. Contrary to Refs. 4 and 5, additional degrees of freedom to take into account transverse normal strain are unnecessary. The implication of this statement is simply that a theory with no more degrees of freedom than are found in so-called first-order shear deformation theory is sufficient if it includes the contributions of the warping displacement to the potential of the applied loads, which are of the same order as terms in the strain energy caused by transverse shear deformation. This is well within the realm of the more classical theories and leads to the expectation that simpler and hence more computationally efficient theories can be used in problems involving contact.

Although recovery relations are published for nonlinear analysis of laminated plates and shells, to present an analytical solution herein the development is limited to the linear theory of isotropic plates. The total potential for isotropic plates is presented in the next section. Results obtained for a loaded plate undergoing cylindrical bending and being pressed against a rigid, smooth surface are used to illustrate the concept.

Total Potential

For small displacements the total potential for a laminated, composite plate can be written as[9]

$$U + V = \tfrac{1}{2}\mathcal{R}^T A \mathcal{R} + \tfrac{1}{2}\gamma^T G \gamma + \mathcal{R}^T F - v^T f - \phi^T m \qquad (1)$$

where U is the strain energy per unit area; V is the potential of applied loads per unit area; $\mathcal{R}$ is a column matrix containing the three membrane measures (ϵ_{11}, $2\epsilon_{12}$, and ϵ_{22}) and the three bending-twist measures (κ_{11}, $2\kappa_{12}$, κ_{22}) of plate deformation; γ is a column matrix containing the two plate transverse shear measures, $2\gamma_{13}$ and $2\gamma_{23}$; v is a column matrix that contains the three displacement components of the plate averaged through the thickness; and ϕ is a column matrix that contains two rotation measures associated with the normal line element. The matrices A, G, and F are thus functions of the number of layers and the material properties of each layer; f, m, and F are also functions of upper and lower surface tractions and body forces. The matrix A is the well-known 6×6 matrix of lamination theory, whereas the others are determined by the variational-asymptotic method. Details are found in Ref. 9.

For the purposes of the present Note, we specialize the theory to the case of isotropic materials and cylindrical bending so that

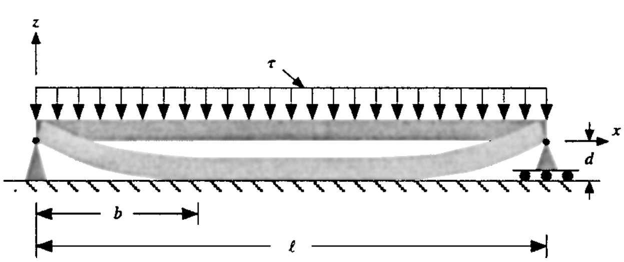

Fig. 1 Schematic of cylindrical bending of a plate, looking along the infinite direction y, contacting a rigid, smooth surface between $b \le x \le \ell - b$; τ is shown as negative, and the contact force per unit area β, acting $b \le x \le \ell - b$, is not shown.

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an analytical solution can be obtained. For the independent variables let x be along the lateral direction such that $0 \le x \le \ell$, y in the longitudinal direction such that $-\infty < y < \infty$, and z through the thickness such that $-h/2 \le z \le h/2$. A rigid, smooth surface parallel to the undeformed plate is at $z = -(d + h/2)$, as shown in Fig. 1. We neglect body forces and introduce upper and lower surface distributed forces per unit area, τ and β, respectively, both positive along the z axis. The total potential then reduces to

$$U + V = \frac{Ehu'^2}{2(1-\nu^2)} + \frac{Ghk(\theta + w')^2}{2} + \frac{Eh^3\theta'^2}{24(1-\nu^2)} - \tau\left[w - \underline{\frac{h\nu u'}{2(1-\nu)}} - \underline{\frac{h^2\nu\theta'}{12(1-\nu)}}\right] - \beta\left[d + w - \zeta^2 + \underline{\frac{h\nu u'}{2(1-\nu)}} - \underline{\frac{h^2\nu\theta'}{12(1-\nu)}}\right] \tag{2}$$

where $u = v_x$ is the average displacement through the thickness in the x direction; $w = v_z$ is the average displacement through the thickness in the z direction; $\theta = \phi_y$ is the average rotation of the normal line element about the y axis; E is the Young's modulus; ν is Poisson's ratio; $G = E/[2(1+\nu)]$; $()'$ denotes a derivative with respect to x; ζ denotes a slack variable similar to that used in Ref. 3; and the shear correction factor is that which is obtained by specializing Ref. 9 to the isotropic case, namely,

$$k = \frac{5(11 - 12\nu + 34\nu^2 - 12\nu^3 + 11\nu^4)}{2(33 - 40\nu + 98\nu^2 - 40\nu^3 + 29\nu^4)} \tag{3}$$

For the problem posed, τ is a specified constant, and $\beta(x)$ is an unknown contact stress stemming from the condition that the bottom of the plate not go below the position $z = -(d + h/2)$. The terms in square brackets contain the effect of the displacements of the upper and lower surfaces and couple the axial displacement and bending to the lateral forces τ and β; β is zero when the plate is not in contact with the surface. The equation governing the slack variable is simply that $\zeta\beta = 0$, so that one of the two must vanish.

Because the displacement u does not appear in the Lagrangian, it is an ignorable coordinate; because there is no applied force along the x direction, a reduced-order Euler–Lagrange equation for u can be found as

$$\frac{Ehu'}{1-\nu^2} + \frac{h\nu(\tau - \beta)}{2(1-\nu)} = 0 \tag{4}$$

so that u' can be eliminated as

$$u' = -\frac{\nu(1+\nu)(\tau - \beta)}{2E} \tag{5}$$

Taking the plate as simply supported at both edges $x = 0$ and $x = \ell$, for $\tau < 0$ and for sufficiently large values of $|\tau|$ the plate will touch the surface for $b < x < \ell - b$ with all deformation symmetric about $x = \ell/2$. The unknowns of the problem then are reduced to w, θ, β, and b. The boundary conditions can be expressed as $w(0) = \partial L/\partial\theta'(0) = 0$ and $\theta(\ell/2) = w'(\ell/2) = 0$. Furthermore, there are internal boundary conditions on the continuity of w, θ, $\partial L/\partial w'$, and $\partial L/\partial\theta'$ at $x = b$. Finally, the constant b is found by setting $\partial L/\partial\beta(b) = 0$.

First we consider the portion of the plate that is not in contact with the surface, in which $\beta = 0$. Here the solution for ζ is simple (and useless). The remaining two Euler–Lagrange equations are

$$Ghk(\theta + w')' + \tau = 0, \qquad \frac{Eh^3\theta''}{12(1-\nu^2)} - Ghk(\theta + w') = 0 \tag{6}$$

and the extent of the contact region is determined by setting

$$\frac{h^2\nu\theta'(b)}{12(1-\nu)} - w(b) = d - \frac{h\nu^2(1+\nu)\tau}{4E(1-\nu)} \tag{7}$$

Polynomial solutions for w and θ are easily obtained in this regime in terms of a single arbitrary constant c_0.

In the contact region ζ is zero, and β is not known. The three Euler–Lagrange equations are

$$\frac{h^2\nu\theta'}{12(1-\nu)} - w + \frac{h\nu^2(1+\nu)(\tau - \beta)}{4E(1-\nu)} = d$$

$$Ghk(\theta + w')' + \tau + \beta = 0$$

$$\frac{Eh^3\theta''}{12(1-\nu^2)} - Ghk(\theta + w') + \frac{h^2\nu\beta'}{12(1-\nu)} = 0 \tag{8}$$

Elimination of β and w leaves a fourth-order equation in θ

$$h^2[h^2k\nu^2\theta^{\text{iv}} - 6(1-\nu)(1-k\nu)\theta''] + 36k(1-\nu)^2\theta = 0 \tag{9}$$

the solution of which is expressible as

$$\theta = c_1e^{-(xs_1/h)} + c_2e^{xs_1/h} + c_3e^{-(xs_2/h)} + c_4e^{xs_2/h} \tag{10}$$

where

$$s_1 = \frac{\sqrt{3}\sqrt{1-\nu}\sqrt{1 - k\nu - \sqrt{1 - 2k\nu - 3k^2\nu^2}}}{\sqrt{k}\nu}$$

$$s_2 = \frac{\sqrt{3}\sqrt{1-\nu}\sqrt{1 - k\nu + \sqrt{1 - 2k\nu - 3k^2\nu^2}}}{\sqrt{k}\nu} \tag{11}$$

Finally, β can be found by integration of

$$\beta' = \frac{E[12k(1-\nu)\theta - h^2(2 - k\nu)\theta'']}{h\nu(1+\nu)(2+3k\nu)} \tag{12}$$

which introduces the arbitrary constant c_5. The six arbitrary constants $c_0, c_1, \ldots, c_5$ can be eliminated by using the two boundary conditions at $x = \ell/2$ and the four internal boundary conditions at $x = b$. The resulting solution is easily obtained by using the symbolic manipulator Mathematica™ but is not presented here because of its length.

The parameters of the problem are defined in terms of nondimensional quantities $\phi = b/\ell$, $\sigma = h/\ell$, $\epsilon = \tau/E$, and $\delta = d/h$. Figure 2 shows the relationship between the normalized contact coordinate ϕ vs the normalized upper surface stress ϵ. Figures 3–5 show vs $\xi = x/\ell$, respectively, the normalized plate shear-stress resultant (same as the transverse shear strain measure) $\theta + w'$, the normalized plate bending stress resultant (same as the normalized bending strain measure) $\ell\theta'$, and the normalized contact stress β/E. The solid lines depict the solution from the present theory, and the dashed lines are the solution based on omitting the underlined terms from V in Eqs. (2), that is, Reissner–Mindlin theory without the terms related to warping of the normal line element, which are of the same order as the correction of U for transverse shear deformation. Only small differences are seen between the results for the contact coordinate. Similarly, differences are small between the results that include warping and those that do not for the displacement w and rotation θ (not shown because of space limitations). The normalized bending strain in Fig. 3 also exhibits small overall quantitative

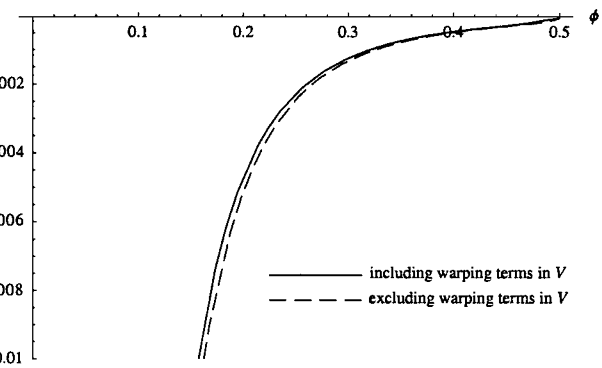

Fig. 2 Normalized upper surface stress $\epsilon = \tau/E$ vs normalized contact coordinate $\phi = b/\ell$ for the case of $\delta = 0.2$, $\sigma = 0.08$, and $\nu = \frac{1}{3}$.

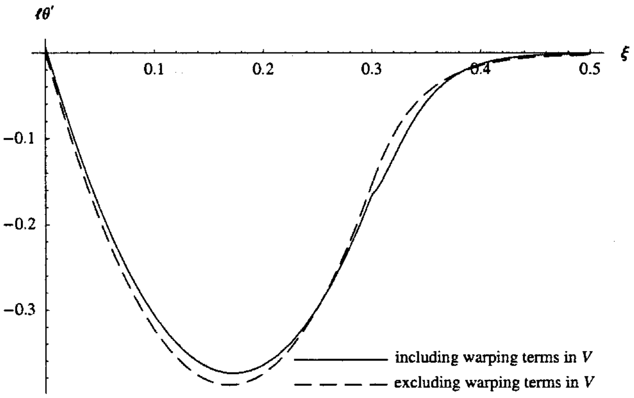

Fig. 3 Normalized bending strain for the case of $\delta = 0.2$, $\sigma = 0.08$, $\epsilon = -0.00123183$, $\phi = 0.3$, and $\nu = \frac{1}{3}$.

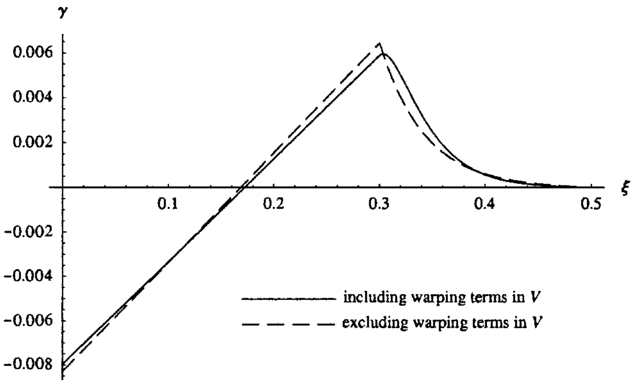

Fig. 4 Transverse shear strain for the case of $\delta = 0.2$, $\sigma = 0.08$, $\epsilon = -0.00123183$, $\phi = 0.3$, and $\nu = \frac{1}{3}$.

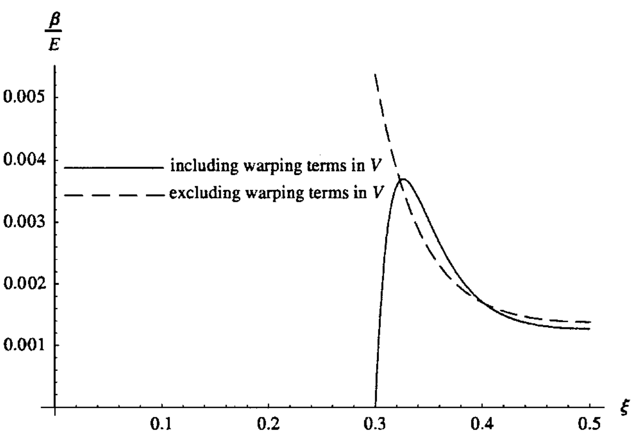

Fig. 5 Normalized contact stress for the case of $\delta = 0.2$, $\sigma = 0.08$, $\epsilon = -0.00123183$, $\phi = 0.3$, and $\nu = \frac{1}{3}$.

differences, though somewhat more noticeable. However, whereas both results in Fig. 4 for the transverse shear are continuous across the value $x = b$, the first derivative is only continuous when the warping terms are retained. From the second equation of Eqs. (8), one sees that the continuity of the first derivative should be reflected in the results for β. Indeed, Fig. 5 clearly shows a jump in the contact stress β when the underlined terms are omitted, whereas results from the theory including the underlined terms are continuous, as they should be. The present results exhibit a stress concentration in the vicinity of the contact boundary. Clearly, the underlined terms allow the present theory to achieve qualitatively correct results for transverse shear resultants and contact stress without any change in the physical phenomena included in the strain energy functional of Reissner–Mindlin theory and without keeping terms of any order higher than those present in Reissner–Mindlin theory. Results for laminated, composite plates and shells should be similarly improved because the published asymptotic theories hold for both. Thus, finite elements based on the complete asymptotically correct theory, with the warping terms in the total potential, should yield results superior to those without them.

Conclusions

The total potential of an isotropic plate is presented for an asymptotically derived plate theory that takes into account the warping of the normal line element but has only the usual Reissner–Mindlin types of deformation in the strain energy functional. A uniformly loaded plate, which is pressed against a rigid, smooth surface and subject to boundary conditions that admit a cylindrical bending solution, is analyzed with this theory. Results obtained exhibit qualitatively correct behavior for transverse shear-stress resultants and transverse normal contact stresses. Thus, contrary to implications in published work, higher-order models with transverse normal strain in the strain energy are not necessary for overcoming the supposed deficiencies in the more classical theories. Rather, continuous contact stresses can be obtained from models that properly take into account warping displacements that are induced by deformation in the work done by applied loads, which terms are asymptotically correct to the same order as Reissner–Mindlin plate theory. This assertion is expected to hold also for composite plates and shells. However, the truth of this assertion, including the quantitative accuracy of the present results, must be ascertained from three-dimensional solutions or experiments.

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